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**EFFECT OF CIRCULAR ISOTROPIC DISK ON THE STRESS
DISTRIBUTION IN AN ORTHOTROPIC PLATE**

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STRESS DISTRIBUTION IN A CIRCULAR ISOTROPIC DISK ON THE STRESS DISTRIBUTION IN AN ORTHOTROPIC PLATE *

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Imagine a thin orthotropic plate of uniform thickness having two perpendicular axes of elastic symmetry in the plane of the plate. An infinite rectangular plate of this type containing a circular isotropic disk is considered. A uniform tension is assumed to act along two opposite edges of the plate. For simplicity the edges of the plate will be taken to be parallel to the two axes of elastic symmetry. It is further assumed that the strains are small and remain within the limits of perfect elasticity. The problem will be treated as a problem of plane stress.

The center of the circular disk is chosen as the center of the coordinate system and the X and Y - axes are chosen to be parallel to the axes of elastic symmetry of the plate. Using the notation of S. Timoshenko¹ the boundary conditions are

$$\left. \begin{aligned} \sigma_x &= S \\ \sigma_y &= 0 \\ \tau_{xy} &= 0 \end{aligned} \right\} r \rightarrow \infty$$

$$\left. \begin{aligned} \sigma_r' &= \sigma_r \\ \tau_{r\theta}' &= \tau_{r\theta} \\ u'(x, y) &= u(x, y) \\ v'(x, y) &= v(x, y) \end{aligned} \right\} r = a \quad (1)$$

* This paper was written under an office of Naval Research contract.

¹ Timoshenko, S., Theory of Elasticity (New York: McGraw Hill Book Company, 1951.)

In the statement of the boundary conditions the prime denotes quantities associated with the disk, and a is the radius of the disk whose boundary is given by:

$$X = a \cos \theta$$

$$Y = a \sin \theta$$

Whenever the word plate is used the region exterior to the disk is intended. In the plate the components of stress and of strain are related as follows²:

$$\begin{aligned} e_x &= \frac{\partial u}{\partial x} = \frac{\sigma_x}{E_x} - \frac{\nu_{yx}}{E_y} \sigma_y \\ e_y &= \frac{\partial v}{\partial y} = \frac{\sigma_y}{E_y} - \frac{\nu_{xy}}{E_x} \sigma_x \\ \gamma_{xy} &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = \frac{1}{G_{xy}} \tau_{xy} \end{aligned} \quad (2)$$

In these relations E_x and E_y are Young's moduli in the X and Y directions respectively; ν is Poisson's ratio; and G is the modulus of rigidity.

For plane stress the equations of equilibrium are:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0 \quad (3)$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0$$

Equations of equilibrium will obviously be satisfied

if we assume a stress function F such that

$$\begin{aligned} \sigma_x &= \frac{\partial^2 F}{\partial y^2} \\ \sigma_y &= \frac{\partial^2 F}{\partial x^2} \\ \tau_{xy} &= -\frac{\partial^2 F}{\partial x \partial y} \end{aligned} \quad (4)$$

² C. Bassel Smith, Quarterly of Applied Math. 6, 452-456 (1949)

Substitution of equations (4) into equations (2), using the relation $\frac{\gamma_{yx}}{E_y} = \frac{\gamma_{xy}}{E_x}$, gives

$$\begin{aligned} e_x &= \frac{1}{E_x} \frac{\partial^2 F}{\partial y^2} - \frac{\gamma_{xy}}{E_x} \frac{\partial^2 F}{\partial x^2} \\ e_y &= \frac{1}{E_y} \frac{\partial^2 F}{\partial x^2} - \frac{\gamma_{yx}}{E_y} \frac{\partial^2 F}{\partial y^2} \\ \gamma_{xy} &= -\frac{1}{G_{xy}} \frac{\partial^2 F}{\partial x \partial y} \end{aligned} \quad (5)$$

If these values are substituted into the compatibility equation,

$$\frac{\partial^2 e_x}{\partial y^2} + \frac{\partial^2 e_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y},$$

one has the following equation

$$\frac{1}{E_y} \frac{\partial^4 F}{\partial x^4} + \left[\frac{1}{G_{xy}} - \frac{2\gamma_{xy}}{E_x} \right] \frac{\partial^4 F}{\partial x^2 \partial y^2} + \frac{1}{E_x} \frac{\partial^4 F}{\partial y^4} = 0$$

$$\text{Let } K = \frac{1}{2} \sqrt{E_x E_y} \left[\frac{1}{G_{xy}} - \frac{2\gamma_{xy}}{E_x} \right], \quad \epsilon = \sqrt[4]{\frac{E_y}{E_x}} \quad (6)$$

and $\eta = \epsilon y$. Then the compatibility equation can be written as

$$\frac{\partial^4 F}{\partial x^4} + 2K \frac{\partial^4 F}{\partial x^2 \partial \eta} + \frac{\partial^4 F}{\partial \eta^4} = 0 \quad (7)$$

Assume $F = F(x + \delta \eta)$ and substitute this into the new form of the compatibility equation to obtain the following equation in δ

$$\delta^4 + 2K\delta^2 + 1 = 0$$

Therefore

$$\delta^2 = -K \pm \sqrt{K^2 - 1}$$

Since a wooden plate will be chosen as an example and, since for wood K as defined is probably always greater than one³, let $K = \cosh \phi$ then $\delta^2 = -\cosh \phi \pm \sinh \phi$ and

$$\begin{aligned}\delta_1 &= i e^{\frac{\phi}{2}} \\ \delta_2 &= i e^{-\frac{\phi}{2}} \\ \delta_3 &= -i e^{\frac{\phi}{2}} \\ \delta_4 &= -i e^{-\frac{\phi}{2}}\end{aligned}$$

Now the compatibility equation can be rewritten as

$$\left(\frac{\partial^2}{\partial x^2} + \alpha^2 \frac{\partial^2}{\partial \eta^2}\right) \left(\frac{\partial^2}{\partial x^2} + \beta^2 \frac{\partial^2}{\partial \eta^2}\right) F = 0 \quad (8)$$

where $\alpha^2 = e^{\phi}$, $\beta^2 = e^{-\phi}$.

It can be shown by direct substitution that the compatibility equation is satisfied by any stress function of the form

$$F = R \left[F_1(x + i\alpha\eta) + F_2(x + i\beta\eta) \right] \quad (9)$$

where

$$\begin{aligned}\alpha &= \sqrt{K + \sqrt{K^2 - 1}} \\ \beta &= \sqrt{K - \sqrt{K^2 - 1}}\end{aligned}$$

Consider an infinite rectangular orthotropic plate containing a circular or elliptic hole. Assume that a uniform tension is applied along two opposite edges and that the edges of the plate

³ C.B. Smith, Effect of Elliptic or Circular Holes on the stress Distribution in plates of wood or plywood considered as orthotropic materials, Forest Products Laboratory Report No 1510, 1944.

are parallel to the two perpendicular axes of elastic symmetry.
(In the case of the elliptic hole choose the axes of the ellipse to be the coordinate axes and assume that these axes are parallel to the elastic axes.) For this problem the solution is known⁴. The stress function is

$$F = R \left[\frac{A}{2\gamma_1^2} \left(\frac{1}{2} (Z_1 - W_1)^2 + \gamma_1^2 \ln(Z_1 + W_1) \right) + \frac{B}{2\gamma_1^2} \left(\frac{1}{2} (Z_1 - W_1)^2 + \gamma_1^2 \ln(Z_1 + W_1) \right) \right] + \frac{1}{2} S_2^2 \quad (10)$$

In this stress function A and B are complex constants and Z_1 , W_1 , γ_1 , α , and β are defined as follows:

$$\begin{aligned} Z_1 &= x + i\alpha\eta & Z_2 &= x + i\beta\eta \\ W_1 &= \sqrt{Z_1^2 - \gamma_1^2} & W_2 &= \sqrt{Z_2^2 - \gamma_2^2} \\ \gamma_1^2 &= a^2(1 - \alpha^2\epsilon^2) & \gamma_2^2 &= a^2(1 - \beta^2\epsilon^2) \\ \alpha &= \sqrt{K + \sqrt{K^2 - 1}} & \beta &= \sqrt{K - \sqrt{K^2 - 1}} \end{aligned}$$

In order that the stresses shall be single-valued, W_1 and W_2 are assigned values such that the following inequalities are always satisfied: $|Z_1 + W_1| \geq \gamma_1$, $|Z_2 + W_2| \geq \gamma_2$.

For the problem of the infinite orthotropic plate containing a circular or an elliptic hole the boundary conditions at infinity are identical to the boundary conditions at infinity for the problem of this paper. From the nature of the problem considered here it appears that the form of the stress function given above will be suitable for the plate. It will be shown that this is true.

⁴ Smith, *ibid*

From the stress function F it follows that

$$\begin{aligned}\sigma_x &= \frac{\partial^2 F}{\partial y^2} = R \left[\frac{\alpha^2 \epsilon^2 A}{W_1(z_1 + W_1)} + \frac{\beta^2 \epsilon^2 B}{W_2(z_2 + W_2)} \right] + S \\ \sigma_y &= \frac{\partial^2 F}{\partial x^2} = R \left[\frac{-A}{W_1(z_1 + W_1)} + \frac{-B}{W_2(z_2 + W_2)} \right] \\ \tau_{xy} &= -\frac{\partial^2 F}{\partial x \partial y} = R \left[\frac{i \alpha \epsilon A}{W_1(z_1 + W_1)} + \frac{i \beta \epsilon B}{W_2(z_2 + W_2)} \right]\end{aligned}\quad (11)$$

The boundary conditions at infinity, namely,

$$\left. \begin{aligned}\sigma_x &= S \\ \sigma_y &= 0 \\ \tau_{xy} &= 0\end{aligned} \right\} \quad r \rightarrow \infty$$

can easily be shown to be satisfied provided only that A and B are finite.

In order to satisfy the boundary conditions at $r=a$, use will be made of the following relationships between polar and rectangular stresses,

$$\begin{aligned}\sigma_r &= \sigma_x \cos^2 \theta + \sigma_y \sin^2 \theta + 2 \tau_{xy} \sin \theta \cos \theta \\ \tau_{r\theta} &= -\sigma_x \sin \theta \cos \theta + \sigma_y \sin \theta \cos \theta + \tau_{xy} (\cos^2 \theta - \sin^2 \theta)\end{aligned}\quad (12)$$

where θ is the polar angle. From these relations using (11) σ_r and $\tau_{r\theta}$ for the plate are found to be

$$\begin{aligned}\sigma_r &= R \left[\frac{A}{W_1(z_1 + W_1)} (\alpha^2 \epsilon^2 \cos^2 \theta - \sin^2 \theta + 2 i \alpha \epsilon \sin \theta \cos \theta) \right. \\ &\quad \left. + \frac{B}{W_2(z_2 + W_2)} (\beta^2 \epsilon^2 \cos^2 \theta - \sin^2 \theta + 2 i \beta \epsilon \sin \theta \cos \theta) \right] \\ &\quad + S \cos^2 \theta. \\ \tau_{r\theta} &= R \left[\frac{-A \{ \sin \theta \cos \theta (1 + \alpha^2 \epsilon^2) + i \alpha \epsilon (\sin^2 \theta - \cos^2 \theta) \}}{W_1(z_1 + W_1)} \right. \\ &\quad \left. + \frac{-B \{ \sin \theta \cos \theta (1 + \beta^2 \epsilon^2) - i \beta \epsilon (\sin^2 \theta - \cos^2 \theta) \}}{W_2(z_2 + W_2)} \right] \\ &\quad - S \sin \theta \cos \theta.\end{aligned}\quad (13)$$

the problem of the stress distribution in a ring or portion of a ring, which includes as a special case a circular disk, is given by S. Timoshenko⁵. To prevent multiple valued stresses and to prevent infinite stresses at the origin some of the constants of the stress function are necessarily zero. From Timoshenko's stress function σ_r' and $\tau_{r\theta}'$ for the disk are found to be

$$\begin{aligned}\sigma_r' = & 2b_0 + 2b_1 r \cos \theta + 2d_1 r \sin \theta \\ & + \sum_{n=2}^{\infty} (na_n r^{n-2} + [n+2] b_n r^n) \cos n\theta \\ & + \sum_{n=2}^{\infty} (nc_n r^{n-2} + [n+2] d_n r^n) \sin n\theta \\ & - \sum_{n=2}^{\infty} (a_n r^{n-2} + b_n r^n) n^2 \cos n\theta \\ & - \sum_{n=2}^{\infty} (c_n r^{n-2} + d_n r^n) n^2 \sin n\theta. \\ \tau_{r\theta}' = & 2b_1 r \sin \theta - 2d_1 r \cos \theta + \sum_{n=2}^{\infty} [(n-1)a_n r^{n-2} \\ & + [n+1]b_n r^n] n \sin n\theta - \sum_{n=2}^{\infty} [(n-1)c_n r^{n-2} \\ & + [n+1]d_n r^n] n \cos n\theta.\end{aligned}\tag{14}$$

If σ_r (13); $\tau_{r\theta}$ (13); σ_r' (14); and $\tau_{r\theta}'$ (14) are evaluated for $r = a$ and if σ_r ($r = a$) is equated to σ_r' ($r = a$) and $\tau_{r\theta}$ ($r = a$) to $\tau_{r\theta}'$ ($r = a$) then the boundary conditions (1) are satisfied except for those relating to displacements.

When the horizontal and vertical displacements in the plate are equated to the horizontal and vertical displacements in the disk respectively as required by the boundary conditions (1)

⁵ Timoshenko, op. cit. P 114

then u and v of the stress function should be uniquely determined. If Timoshenko's stress function for the disk is expressed as a function of X and Y and the rectangular stresses are obtained by differentiation then the rectangular displacements in the disk can be obtained by substitution of the expressions for the rectangular stresses into equations (2) and integrating. This yields⁶

$$\begin{aligned} u' &= \frac{1}{E} \left\{ [2b_0(1-\nu) - 2a_0(1+\nu)]x + 6(1-\nu)d_1x^2y \right\} + u'_0(y) \\ v' &= \frac{1}{E} \left\{ [2b_0(1-\nu) + 2a_0(1+\nu)]y + 6(1-\nu)d_1xy^2 \right\} + v'_0(x) \end{aligned} \quad (15)$$

$u'_0(y)$ and $v'_0(x)$ can be shown to be equal to zero.

The rectangular stresses for the plate are known (11). In a similar manner the displacements in the plate are found to be

$$\begin{aligned} u &= R \left[\frac{A(\alpha^2 \epsilon^2 + \nu_{yx})}{E_x} \frac{-1}{z_1 + w_1} + \frac{B(\beta^2 \epsilon^2 + \nu_{xy})}{E_x} \frac{-1}{z_2 + w_2} \right] \\ &\quad + \frac{Sx}{E_x} + u_0(y) \\ v &= R \left[\frac{A(1 + \alpha^2 \epsilon^2 \nu_{yx})}{E_y} \frac{1}{i\alpha\epsilon(z_1 + w_1)} + \frac{B(1 + \beta^2 \epsilon^2 \nu_{xy})}{E_y} \frac{1}{i\beta\epsilon(z_2 + w_2)} \right] \\ &\quad - \frac{Sy}{E_y} + v_0(x) \end{aligned} \quad (16)$$

where $u_0(y)$ and $v_0(x)$ can be shown to be equal to zero.

Equating U (16) to U' (15) and V (16) to V' (15) imposes

⁶ Since the disk is isotropic $E_x = E_y = E$ and $\nu_{xy} = \nu_{yx} = \nu$

the last two boundary conditions (1). This determines A and B uniquely.

$$A = \frac{\left[\frac{(E-E_v) \alpha^2 S}{(\alpha E - \nu) E_x + (\alpha^2 E^2 + \nu_{xy}) E} \right] \left[\frac{\alpha(1+\alpha E) [(\beta E - \beta^2 E^2 \nu) E_y + (1+\beta^2 E^2 \nu_{yx}) E]}{\beta(1+\beta E) [(\alpha E - \alpha^2 E^2 \nu) E_y + (1+\alpha^2 E^2 \nu_{yx}) E]} \right]}{\frac{\alpha [\beta E(1-\beta E \nu) E_y + (1+\beta^2 E^2 \nu_{yx}) E]}{\beta(1+\beta E) [\alpha E(1-\alpha E \nu) E_y + (1+\alpha^2 E^2 \nu_{yx}) E]} - \frac{(\beta E - \nu) E_x + (\beta^2 E^2 + \nu_{xy}) E}{(\beta E + 1) [(\alpha E - \nu) E_x + (\alpha^2 E^2 + \nu_{xy}) E]}} \quad (17)$$

$$B = \frac{\left[\frac{\alpha E(1+\alpha E)(\nu E_y - \nu_{yx} E) \alpha^2 S}{(\alpha E - \alpha^2 E^2 \nu) E_y + (1+\alpha^2 E^2 \nu_{yx}) E} \right] \left[\frac{(\beta E - \nu) E_x + (\beta^2 E^2 + \nu_{xy}) E}{(\beta E + 1) [(\alpha E - \nu) E_x + (\alpha^2 E^2 + \nu_{xy}) E]} \right]}{\frac{\alpha [\beta E(1-\beta E \nu) E_y + (1+\beta^2 E^2 \nu_{yx}) E]}{\beta(1+\beta E) [\alpha E(1-\alpha E \nu) E_y + (1+\alpha^2 E^2 \nu_{yx}) E]} - \frac{(\beta E - \nu) E_x + (\beta^2 E^2 + \nu_{xy}) E}{(\beta E + 1) [(\alpha E - \nu) E_x + (\alpha^2 E^2 + \nu_{xy}) E]}} \quad (18)$$

$$B = \frac{\left[\frac{\alpha E(\nu E_y - \nu_{yx} E) \alpha^2 S}{\alpha E(1-\alpha E \nu) E_y + (1+\alpha^2 E^2 \nu_{yx}) E} \right] - \left[\frac{(E-E_x) \alpha^2 S}{(\alpha E - \nu) E_x + (\alpha^2 E^2 + \nu_{xy}) E} \right]}{\frac{\alpha [\beta E(1-\beta E \nu) E_y + (1+\beta^2 E^2 \nu_{yx}) E]}{\beta(1+\beta E) [\alpha E(1-\alpha E \nu) E_y + (1+\alpha^2 E^2 \nu_{yx}) E]} - \frac{(\beta E - \nu) E_x + (\beta^2 E^2 + \nu_{xy}) E}{(\beta E + 1) [(\alpha E - \nu) E_x + (\alpha^2 E^2 + \nu_{xy}) E]}}$$

It is desirable to show that for a rigid disk, that is, for $E = \infty$, the stress function of this paper reduces to the known stress function for the case of the rigid disk⁷. It is sufficient to show that the constants A (17) and B (18) reduce to the proper form since the form of the stress functions for these two cases is the same. By taking the limit of A and of B as $E \rightarrow \infty$ and simplifying it follows that

⁷ Alvin Jewel Owens "Effect of a Rigid Elliptic Disk on the Stress Distribution in an Orthotropic Plate." Doctor's Thesis, U. of Florida, 1950.

It is necessary but not sufficient that $E_x = E_y$ for the isotropic case. A sufficient condition is that $\phi \rightarrow 0$ since $K \rightarrow 1$ and (7) reduces to the biharmonic equation for the isotropic case.

The expression for the orthotropic stress function becomes indeterminate as $\phi \rightarrow 0$. Successive application of L'Hospital's rule gives for the isotropic stress function

$$F = \frac{S}{2} R \left\{ \frac{[(2E'+E)[(1+\nu')E - (1+\nu)E'] + (E'-E)[(2+\nu')E + (5-\nu)E']}{[(1-\nu')E + (1+\nu)E'][(1+\nu')E + (3-\nu)E']} \right. \\ \left. + \frac{(\nu'E - \nu E')[\nu'E + (1-\nu)E']}{[(1-\nu')E + (1+\nu)E'][(1+\nu')E + (3-\nu)E']} \right\} a^2 \log 2x \quad (21) \\ + \frac{2[(1+\nu)E' - (1+\nu')E]}{(1+\nu')E + (3-\nu)E'} \left(\frac{ia^2 y}{2} + \frac{a^2}{4x^2} \right) + \frac{S y^2}{2}$$

It can be shown that the stress function reduces to the one given by Timoshenko⁹ for an infinite rectangular orthotropic plate with a circular hole whenever $E' \rightarrow 0$.

As examples of the results obtained, consider a large rectangular, plain-sawn plate of Sitka spruce, and a large rectangular plate of steel. Each plate will be taken to have a small circular copper disk at its center.

Since shearing stresses are probably more important in producing failure in a wooden plate, Figure 1 is given to show the variation of the shear stress component around the boundary of the disk. Also since it is failure of the wooden plate that is of the most concern, stresses along the coordinate axes exterior to the copper disk are given in subsequent figures.

⁹ Timoshenko, op. cit. p 77

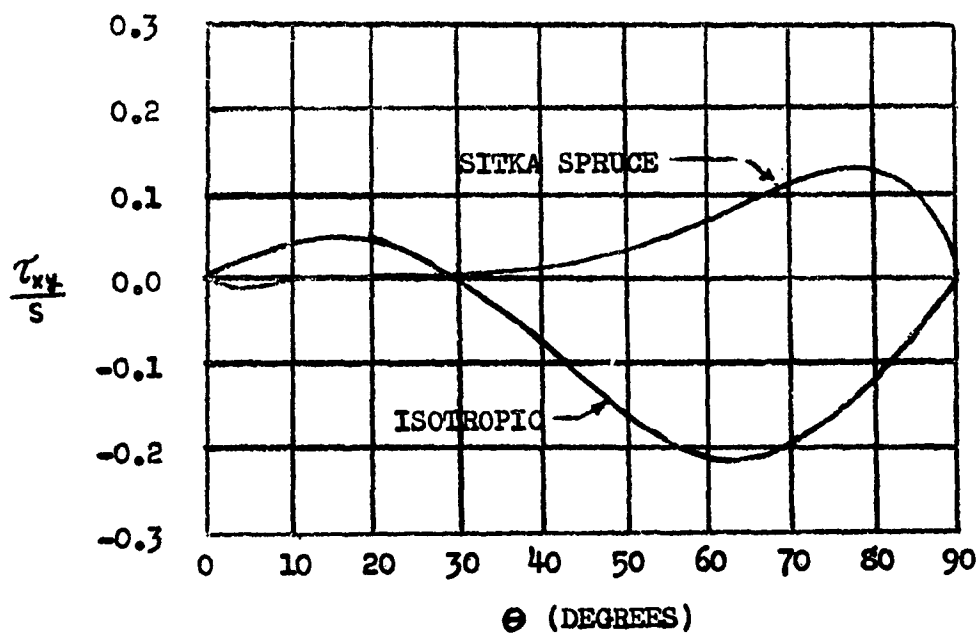


Figure 1. Variation of the shear stress component τ_{xy} at points along the boundary of a circular copper disk of radius a with center at the origin for a plain-sawn plate of Sitka spruce and for an isotropic (steel) plate.

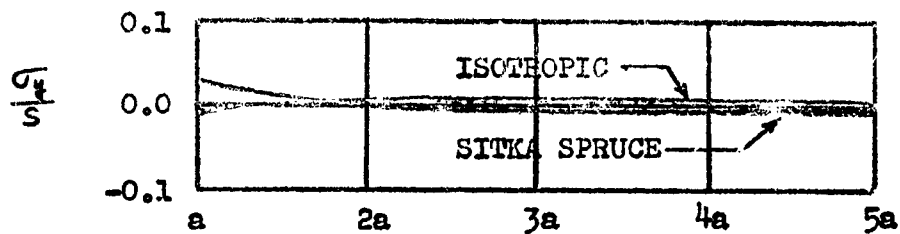


Figure 2. Variation of the normal stress component σ_y at points along the X-axis exterior to a circular copper disk of radius a with center at the origin for a plain-sawn plate of Sitka spruce and for an isotropic plate.

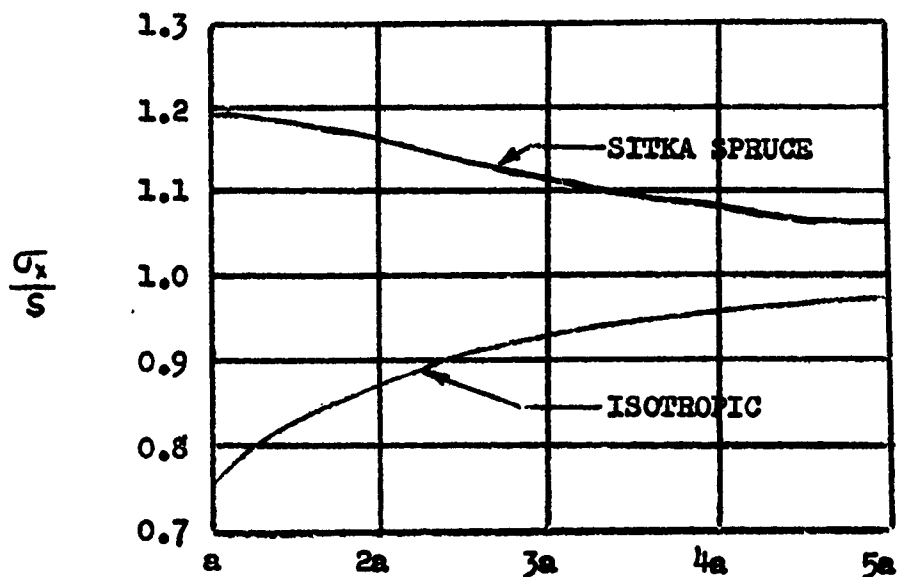


Figure 3. Variation of the normal stress component σ_x at points along the X-axis exterior to a circular copper disk of radius a with center at the origin for a plain-sawn plate of Sitka spruce and for an isotropic plate.

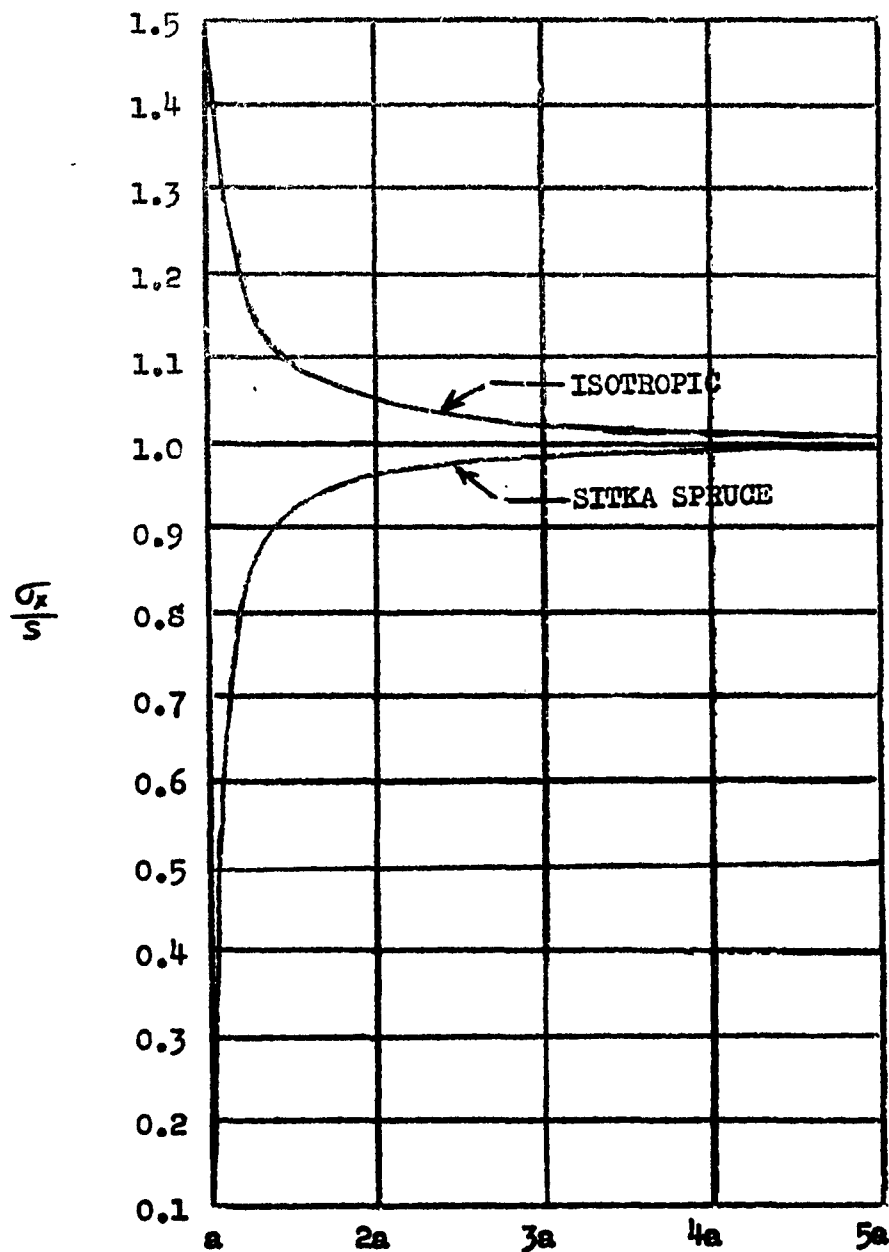


Figure 4. Variation of the normal stress component σ_x at points along the Y-axis exterior to a circular copper disk of radius a with center at the origin for a plain-sawn plate of Sitka spruce and for an isotropic plate.

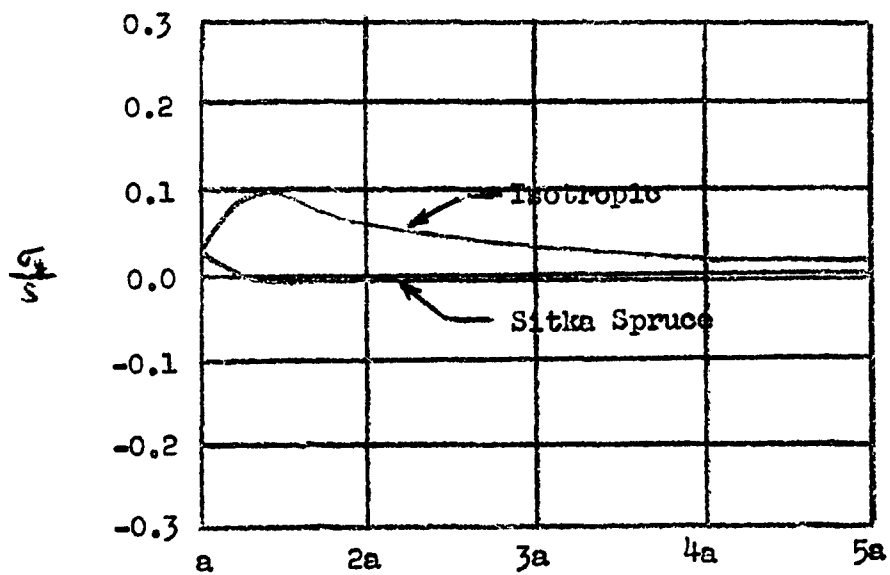


Figure 5. Variation of the normal stress component σ_y at points along the Y-axis exterior to a circular copper disk of radius a with center at the origin for a plain-sawn plate of Sitka spruce and for an isotropic plate.